A generalization of Nesbitt's Inequality **February 2010**

1840. *Proposed by Tuan Le, 12th grade, Fairmont High School, Anaheim, CA.*

Let a, b, and c be nonnegative real numbers such that no two of them are equal to zero. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3\sqrt[3]{abc}}{2(a+b+c)}\geq 2.$$

Solution by Arkady Alt, San Jose, CA.

Let us assume first that one of the numbers is zero. Suppose that $a=0$, then $b,c\neq 0$ and the inequality becomes $b/c+c/b\geq 2$ which is true by the Arithmetic Mean–Geometric Mean Inequality. From now on, assume that $abc\neq 0$. By the Cauchy–Schwarz Inequality,

$$\left(\frac{a^2}{ab+ca}+\frac{b^2}{bc+ab}+\frac{c^2}{ca+bc}\right)((ab+ca)+(bc+ab)+(ca+bc))$$
$$\geq (a+b+c)^2,$$

thus,

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a^2}{ab+ca}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Therefore, it suffices to prove that

$$\frac{(a+b+c)^2}{2(ab+bc+ca)}+\frac{3\sqrt[3]{abc}}{2(a+b+c)}\geq 2\,.$$

This is equivalent to

$$(a+b+c)^3+3\sqrt[3]{abc}(ab+bc+ca)-4(a+b+c)(ab+bc+ca)\geq 0\,,$$

and further equivalent to

$$\left((a+b+c)^3-4(a+b+c)(ab+bc+ca)+9abc\right)$$
$$+3\sqrt[3]{abc}\left(ab+bc+ca-3\sqrt[3]{a^2b^2c^2}\right)\geq 0. \qquad (1)$$

By Schur's inequality $\sum_{cyc}a(a-b)(a-c)\geq 0$. This inequality can be rewritten in the form $(a+b+c)^3\geq 4(a+b+c)(ab+bc+ca)-9abc$, so the expression in the first line of Inequality (1) is nonnegative. By the Arithmetic Mean–Geometric Mean Inequality, $ab+bc+ca\geq 3\sqrt[3]{a^2b^2c^2}$, so the expression in the second line of Inequality (1) is also nonnegative. This completes the proof. Equality is achieved if and only if $a=b=c$ or two of a, b, and c are equal to each other and the other is equal to 0.

Editor's Note. Some of the solutions submitted used either Schur's inequality or a combination of Schur's inequality with Muirhead's Theorem. Michael Vowe makes a substitution that transform the inequality for the positive numbers a, b, and c, into an inequality for the sides of a triangle. Then he reduces this inequality to the well-known geometric inequality $R\geq 2r$, where R is the circumradius of the triangle and r is the inradius.